A Joyful Walk Through Analytic Number Theory: from Classical to Modern

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EPTMT Guest Lecture
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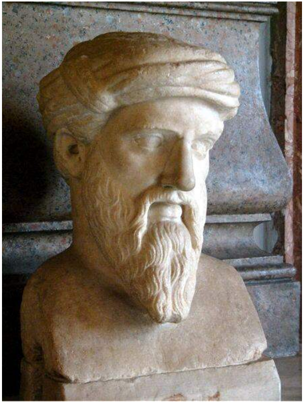
Story Time!

- Many good stories start with ‘once upon a time’.
- Number Theory is one of these stories.
- More precisely, it started from the ancient Greek.
- It was influenced heavily by the philosophy of that era. Interestingly or bizaarely, they thought of the meanings of numbers a bit too hard. (More like numerology)

Perfect Numbers and Pythagoras



- Pythagoreans equated the ‘perfect number’ 6 to marriage, health, and beauty of integrity and agreement.

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- 'Six is a perfect number in itself, and not because God created all things in six days; rather the converse is true. God created all things in six days because the number six is perfect.'

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- Alcuin of York claimed that the Second Origin is not perfect because...
- 'Eight is not a perfect number (in fact deficient) and the number eight stands for the eight souls in Noah's Ark: Noah, his three sons and their four wives. The entire human race was originated from these eight souls.'

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- Please don't ask me further about the philosophical or biblical aspects of perfect numbers. I know nothing about these!

OK, Be Serious Now! Background...

Definition (Perfect Number)

A **perfect number** is a natural number that is equal to sum of all of its proper divisors.

- **Example:** $6 = 1 + 2 + 3$; $28 = 1 + 2 + 4 + 7 + 14$

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- Why writing in σ? Because σ is a **multiplicative** function, i.e., if m, n are relatively prime natural numbers, then $\sigma(mn) = \sigma(m)\sigma(n)$.

Background—What were known?

Theorem (Euler/Euclid)

A natural number is an even perfect number if and only if it is of the form $2^{p-1}(2^p - 1)$, where p is a prime such that $2^p - 1$ is also a prime.

Computational evidence:

- An odd perfect number must be greater than 10^{1500}, has at least 101 prime factors and at least 10 distinct prime factors. The largest prime factor is greater than 10^8. (c.f. Math. Comp. Journal for more!).

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- Does odd perfect number exist?
- And what are you waiting for ?!

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- 3. **Euclid's Theorem** There are infinitely many primes: 2,3,5,7,11,13,17,19,...

Sieve of Eratosthenes

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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- We can count things either roughly or precisely. First roughly...like probability

Definition (Natural Density)

Let $A \subset \mathbb{N}$ be a set. Then A is said to be having natural density d if

$$\lim_{x \to \infty} \frac{\#(A \cap [1, x])}{x} = d.$$

A Second Step

- Say if the set A has density 0, the size of $\#(A \cap [1, x])$ can be

$$\approx \quad \frac{x}{\log x}, \quad \frac{x}{(\log x)^2}, \quad x^{1/3}, \quad \log\log x, \dots$$

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- We want to establish an upper bound to rule out some of its possibilities: $\#(A \cap [1, x]) \leq f(x)$, where $f : [1, \infty) \to [0, \infty)$ is some increasing function such that $f(x) \leq x$.

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- Say if $\#(A \cap [1, x]) \ \leq \ x^{1/2}$, then it is impossible for $A \cap [1, x]$ to have the size $x/(\log x)$, $x/(\log x)^2$,..., but it is possible for it to have size $x^{1/3}$, $\log \log x$,...

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- Statistical behaviour...

Avenue of Primes

- The analytic study of prime numbers began at 18th-19th century by Euler (1707-1783), Gauss (1777-1855), Chebyshev (1821-1894), Dirichlet (1805-1859) and Riemann (1826-1866).

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- It was more fortunate that in the case of prime there were more tools to study them, like complex analysis, sieve methods, circle methods, **but not without limitations**.
- A great achievement in the history in analytic number theory:

Theorem (Prime Number Theorem; Hadamard, de la Vallee-Poussin 1896)

Denote the number of primes from 1 to x by $\pi(x)$, i.e., $\pi(x) = \sum_{p\leq x} 1$. Then we have

$$\pi(x) \sim \frac{x}{\log x}.$$

A first attempt—Chebyshev

- It is better to give some weight during the counting and primes love (natural) logarithm. I would say this is one of the most ingenious part in the development of prime counting!

Definition (von Mangoldt's Function)

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \text{ for some } k \geq 1, \text{ prime } p \\ 0 & \text{otherwise} \end{cases}$$

Well Begun Is Half Done!

- Then we want to estimate the Chebyshev's ψ function, which is defined as follows:

Definition (Chebyshev's Prime Seed)

$$\psi(x) \ := \ \sum_{n \le x} \Lambda(n).$$

Why $\psi(x)$ is so nice for Chebyshev? Mathematical Appreciation...

Really due to several ingenious observations:

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Golden Rule in Analytic Number Theory

Analytic number theorists evaluate a sum by expanding into two or more sums, not to intimate the readers, but to open up possibilities to switch the order of summation signs.

Triple counting

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$$S(x) \;=\; \sum_{m \le x} \psi\left(\frac{x}{m}\right).$$

He was then able to prove that there exists constants $C_1, C_2 > 0$ such that

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This is equivalent to

$$C_1' \frac{x}{\log x} \;<\; \pi(x) \;<\; C_2' \frac{x}{\log x}.$$

More from Chebyshev...

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- (Betrand's postulate) There is a prime between x and $2x$.
- Really remarkable!

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- However, it was an era before calculators or computers. He nevertheless verified this conjecture from 2 to 3×10^6 and he was convinced that it must be true (and hence 'postulate').
- My highest respect to his **perserverance** !

Words from the Wise: Take I

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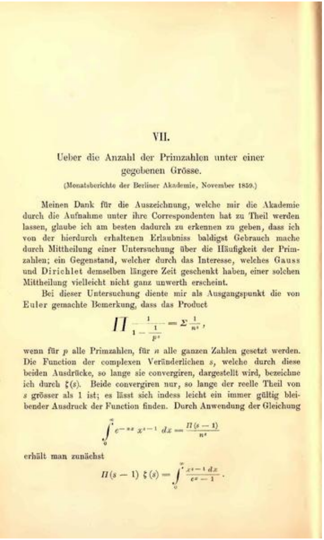
- He is one of the first mathematicians who gave the complete proof of the Prime Number Theorem.

Riemann's Big Bang (1859)



VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\,\zeta(s) = \int_0^\infty \frac{x^{s-1}\,dx}{e^x - 1}.$$

- *On the Number of Primes Less Than a Given Magnitude.*
- His only number theory paper
- Moving from real to complex in the quest of prime counting.

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- 1. It has a nice generating function as Dirichlet's series:

$$-\frac{-\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \text{ for } \Re s > 1,$$

where $\zeta(s)$ is the Riemann zeta function which is defined by

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$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ ,\zeta(6) = \frac{\pi^6}{945}, \dots \text{ (Euler)}$$

Power of Integration by Parts

By thinking of the infinite sum as an integral, we can perform a 'formal' integration by parts:

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_1^\infty \frac{1}{x^s} d\psi(x) = \left[\frac{\psi(x)}{x^s}\right]_1^\infty - s\int_0^\infty \frac{\psi(x)}{x^s}\frac{dx}{x}$$
$$= -s\int_0^\infty \frac{\psi(x)}{x^s}\frac{dx}{x}.$$

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By a standard inversion technique, we have

$$\psi(x) = \frac{1}{2\pi i}\int_{(c)} \frac{-\zeta'(s)}{\zeta(s)}\frac{x^s}{s}dx \quad (\text{ roughly...})$$

Key Points ...

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- 1. $\zeta(s) \neq 0$ on $\Re s = 1$.
- 2. so that the only 'blow-up' (i.e., pole) of $-\zeta'(s)/\zeta(s)$ near $s = 1$, which corresponds to the main term x in PNT (good!) and we can 'capture' it.

$$\zeta(s) \;=\; \frac{1}{s-1} + \gamma + A_1(s-1) + \cdots .$$

More Advantages

- 2. Its estimate can be converted easily to that for $\pi(x)$, trivially by integration by parts. There are other possibilities, but showing that estimate is equivalent to PNT is very tricky!

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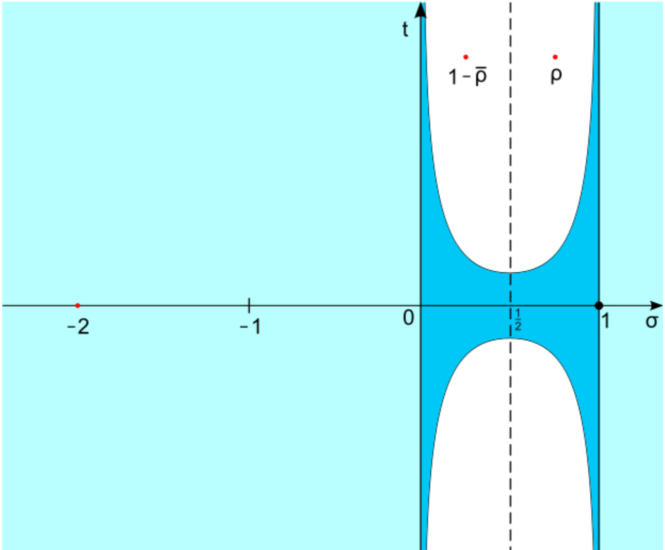
- 2. Its estimate can be converted easily to that for $\pi(x)$, trivially by integration by parts. There are other possibilities, but showing that estimate is equivalent to PNT is very tricky!
- 3. Deeper: it has a simple **explicit formula** which relates $\psi(x)$ to the critical zeros of $\zeta(s)$. You can think of it as an **exact** form of PNT. From this, we can prove that for $\theta \in [1/2, 1)$,

$$\psi(x) \;=\; x + O(x^{\theta+\epsilon})$$

if and only if

$$\zeta(s) \neq 0 \quad \text{for } \Re s > \theta.$$

Analytic Number Theory is Always Alive! Riemann Hypothesis...



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Theorem (Euler Product; Analytic Form of Fundamental Theorem)

$$\zeta(\sigma) = \prod_p \left(1 - \frac{1}{p^\sigma}\right)^{-1} \text{ for } \sigma > 1.$$

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$$\zeta(\sigma) = \prod_p \left(1 - \frac{1}{p^\sigma}\right)^{-1} \text{ for } \sigma > 1.$$

- Euler was able to use it to re-prove there are infinitely many primes:

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- Famous computational number theorist Hendrik W. Lenstra, Jr. once said that "a math talk without a proof is just like a movie without love scene."
- So, it is important to include at least one proof (at least by intimidation!) in every math talk and here we go.

Euler's Proof

Suppose there are only finitely many primes. Then RHS of Euler Product Theorem is a **finite** product. Letting $\sigma \to 1$, we have RHS $\to \prod_p \left(1 - \frac{1}{p}\right)^{-1}$.
Whereas for the LHS, $\sum_{n=1}^{\infty} \frac{1}{n^\sigma} \to \infty$ as $\sigma \to 1$ as it is well-known that the harmonic series diverges. Contradiction!

Euler Take 2

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When you see product, try to take (natural) logarithm!

Also note that $-\log(1-x) \approx x$ for $0 < x < 1/2$. Hence for $\sigma > 1$, we have

$$\log \zeta(\sigma) = -\sum_p \log\left(1 - \frac{1}{p^\sigma}\right) \approx \sum_p \frac{1}{p^\sigma}.$$

Euler Take 2

In fact, Euler had done somewhat better than Euclid. He proved $\sum_p \frac{1}{p} = \infty$ again by using his Product Theorem.

Golden Rule in Analytic Number Theory

When you see product, try to take (natural) logarithm!

Also note that $-\log(1-x) \approx x$ for $0 < x < 1/2$. Hence for $\sigma > 1$, we have

$$\log \zeta(\sigma) = -\sum_p \log\left(1 - \frac{1}{p^\sigma}\right) \approx \sum_p \frac{1}{p^\sigma}.$$

As $\sigma \to 1$, $\zeta(\sigma) \to \infty$ and so $\log \zeta(\sigma) \to \infty$. Therefore, $\lim_{\sigma\to 1} \sum_p \frac{1}{p^\sigma} = \infty$. Since $\frac{1}{p} > \frac{1}{p^\sigma}$, we have $\sum_p \frac{1}{p} = \infty$. We are done.

Final Comments Before Moving On

- Dirichlet modeled on Euler's proof to prove that there are infinitely many primes in arithmetic progression $an + b$ for $(a, b) = 1$. In analog to $\zeta(s)$, he introduced the Dirichlet's L-function $L(s, \chi)$ to select those primes in A.P.

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- Brun (1885-1978) studied the reciprocal sum of **twin primes**. If it diverges, then the well-known **Twin Prime Conjecture** would be solved. It turns out that we have a convergent sum (no conclusion!). However, this led to the birth of **sieve theory**, which dominated the additive, analytic number theory of primes for the past century and we can still see its ripple now.

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- What is sieve theory? Basically we want to make the Sieve of Eratosthenes 'usable' in giving good estimates.

What are primes?

- Algebra: Prime ideal—Prime ideal theorem
- Function field: irreducible polynomials
- Hyperbolic geometry: primitive hyperbolic conjugacy class (geodesic cycle)—Selberg's Prime Geodesic Theorem, Selberg's Zeta Function and Trace Formula
- Graph theory—Ihara zeta function
- Many more...

- A. Granville, Primes in Intervals of Bounded Length
- Y. Motohashi, The Twin Prime Conjecture
- K. Soundararajan, Small Gaps Between Prime Numbers: The Work of Goldston-Pintz-Yildirim
- J. Maynard, Small Gaps Between Primes
- K.Ford, Sieve Lecture Notes

Now Comes to Perfect Numbers

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- The analytic study of perfect numbers was somewhat less fortunate, **but not without progress**.
- Even counting even perfect number is hard: we are counting primes p with an extra, difficult condition $2^p - 1$ is also prime.
- One has to admit that the complex analytic techniques are not flexible enough and are not able to directly attack the counting problems of many arithmetic notions. (Generating functions?!)

A Bit of Notations...

- Big O-notations are used throughout literature in analytic number theory. It simply means we don't care about the constant multiple: we write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ if there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for large x.
- $o(1)$ denotes a quantity tends to 0.

Analytic Progress (Courtesy of P. Pollack)

Let $V(x)$ be the number of perfect numbers up to x. As $x \to \infty$,

- (Davenport 1933) The set of perfect numbers has density 0.

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Conjecture

As $x \to \infty$,

$$V(x) \sim \frac{e^{\gamma}}{\log 2} \log \log x$$

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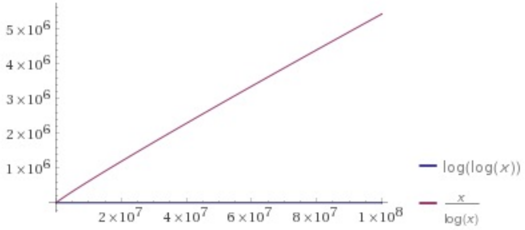
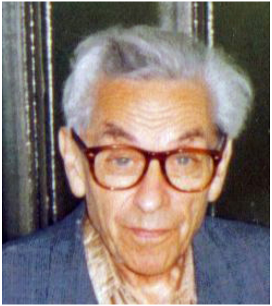
- This quantitative conjecture also suggests why counting perfect numbers is hard: it asymptotic order is like $\log\log x$, which is very very very... slow growing, showing that perfect numbers are very very very... rare.
- For prime numbers, its asymptotic order is $x/\log x$. Although it has density 0 (Alright it is rare to have primes), it is still much more abundant than perfect numbers!

Plot:



How to Approach These Problems Then? Back to Basic...



- Erdös (1913-1996) is one of the pioneers to approach distribution problems of many arithmetic notions by very diverse, beautiful elementary methods and incredible observations.

What had Erdös looked at?

1. Perfect numbers
2. Carmichael numbers
3. Normal orders
4. Image of Euler's totient functions
5. How often does $\sigma(m) = \phi(n)$, $\phi(n) = \phi(n+k)$, $\tau(n) = \tau(n+k)$?
6. Amicable numbers
7. Sociable numbers
8. Multiplication table

The list goes on...

Erdös' idea:

- To give a non-trivial upper bound, you don't have to capture **exactly** the numbers you want to count, but instead **essential features** they satisfy.

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- Of course, you will over-count, but nonetheless it is easier to count and is non-trivial (surprising others)!
- For lower bound, most of the time it is harder as you need some good construction of the numbers you want to count. Usually need lots of ingenuity...

Example: Work of Pomerance

- In 1975, Pomerance studied the distribution of $S_{\ell,k} = \{n \in \mathbb{N} : \sigma(n) = \ell n + k\}$, where $\ell, k \in \mathbb{Z}$, $\ell \geq 2$,

Example: Work of Pomerance

- In 1975, Pomerance studied the distribution of $S_{\ell,k} = \{n \in \mathbb{N} : \sigma(n) = \ell n + k\}$, where $\ell, k \in \mathbb{Z}$, $\ell \geq 2$, which generalizes the following notions:
- $S_{2,0}$ (Perfect numbers),
- $S_{\ell,0}$ (ℓ-multiply perfect numbers),
- $S_{2,1}$ (Quasiperfect numbers),
- $S_{2,-1}$ (Almost perfect numbers)

Theorem (Pomerance 1975)

Denote $S_{\ell,k} \cap [1, x]$ by $S_{\ell,k}(x)$. As $x \to \infty$,

$$\#S_{\ell,k}(x) \ll_k \frac{x}{\log x}$$

In particular, $S_{\ell,k}$ must have density 0.

- Clearly $\sigma(n) = \ell n + k$ implies $\sigma(n) \equiv k \pmod{n}$.

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- To obtain upper bound for $\#S_{\ell,k}(x)$, Pomerance instead counted $n \leq x$ such that $\sigma(n) \equiv k \pmod{n}$.
- Although we lose arithmetic information by reducing $\pmod{n}$, this makes the counting much easier. Of course the bound may not be sharp.

Definition (Regular/ Sporadic)

n is said to be a **regular solution** of $\sigma(n) \equiv k \pmod{n}$ if $n = pm$, with p being a prime, $p \nmid m$, $m \mid \sigma(m)$, and $\sigma(m) = k$. Otherwise n is said to be a **sporadic solution**.

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- It is clear to see that regular solutions are actually solutions!
- Pomerance (1975) observed that we only need to count regular solutions!

Pomerance's Theorem

Theorem (Pomerance 1975)

The number of sporadic solutions of the congruence $\sigma(n) \equiv k \pmod{n}$ is $O_k(x \exp(-\beta(\log x \log\log x)^{1/2}))$ as $x \to \infty$ for any $\beta < 1/\sqrt{2}$.

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- Indeed if $n \in S_{\ell,k}$ and n is of the form of regular solutions, then
$$(1+p)k = \sigma(p)\sigma(m) = \sigma(n) = \ell pm + k.$$
This implies $\sigma(m) = k = \ell m$. So, $m = k/\ell$ is an ℓ-perfect number.

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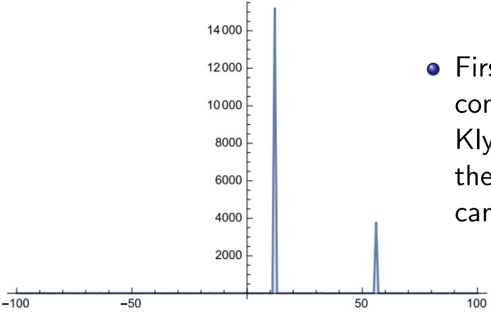
- So if k/ℓ is ℓ-perfect, then we have $\asymp x/\log x$ many regular solutions up to x by the Prime Number Theorem. The number of sporadic solutions up to x is negligible.
- If k/ℓ is not ℓ-perfect, then there is no regular solution. The upper bound for $S_{\ell,k}(x)$ is given by the sporadic bound.

Bias...



- First observed computationally by Klyve-Davis-Kraght in their *Involve* article, but cannot explain why.

A Lemma from Us

Lemma (Cohen-Cordwell-Epstein-K.-Lott-Miller)

For fixed integers k, ℓ with $\ell \geq 2$, as $x \to \infty$, we have

1. If k/ℓ is an ℓ-perfect number, then

$$\#S(\ell, k; x) \ \sim \ \frac{\ell}{k}\frac{x}{\log x}.$$

2. If k/ℓ is not an ℓ-perfect number, then

$$\#S(\ell, k; x) \ \leq \ x^{1/2+o(1)}.$$

In the case of ℓ is even and k is odd, the upper bound can be replaced by $|k|x^{1/4+o(1)}$.

Pomerance Take Two— uniformity and better bound

Theorem (Pollack-Shevelev 2012)

Uniformly for $|k| < x^{2/3}$, the number of sporadic solutions of the congruence $\sigma(n) \equiv k \pmod{n}$ up to x is at most $x^{2/3+o(1)}$, as $x \to \infty$.

Theorem (Anavi-Pollack-Pomerance 2012)

Uniformly for $|k| \leq x^{1/4}$, the number of sporadic solutions of the congruence $\sigma(n) \equiv k \pmod{n}$ up to x is at most $x^{1/2+o(1)}$, as $x \to \infty$.

Fresh from the Oven...

Definition (Modified Regular/ Sporadic)

n is said to be a **regular solution** of $\sigma(n) = \ell n + k$ if $n = pm$, with p being a prime, $p \nmid m$, $\sigma(m) = \ell m$, and $\sigma(m) = k$. Otherwise n is said to be a **sporadic solution**.

Theorem (Pollack-Pomerance-Thompson 2017)

Let ℓ, k be integers with $\ell > 0$. Then the number of sporadic solutions $n \leq x$ of $\sigma(n) = \ell n + k$ is at most $x^{3/5+o_\ell(1)}$ as $x \to \infty$, uniformly in k.

- Much better range in terms of k, but loses uniformity in ℓ and worse upper bound. (Trade-off!).

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What we want to look at?

Definition 4.1 (($\ell; k$)-within-perfect numbers)

Let $\ell > 1$ be a real number, $k : [1, \infty) \to \mathbb{R}$ be an increasing, positive function. n is said to be $(\ell; k)$-within-perfect if $|\sigma(n) - \ell n| < k(n)$. We denote the set of $(\ell; k)$-within-perfect numbers by $W(\ell; k)$ and $W(\ell; k; x) := W(\ell; k) \cap [1, x]$.

- Wolke and Harman studied in terms of a Diophantine approximation.

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- Wolke and Harman studied in terms of a Diophantine approximation.
- They showed that for any real $\ell \geq 1$ and for any $c \in (0.525, 1)$, there exists infinitely many natural numbers that are $(\ell; y^c)$-within-perfect.

Distribution

Definition (Distribution function)

An arithmetic function $f : \mathbb{N} \to \mathbb{R}$ has a distribution function if there exists an increasing, continuous function $F : (a, b) \to \mathbb{R}$ such that

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : f(n) \leq u\} = F(u).$$

In terms of asymptotic densities...Phase Transition!

Theorem(Cohen-Cordwell-Epstein-K.-Lott-Miller)

Let $D(\cdot)$ denote the distribution function of $\sigma(n)/n$. (by Davenport 1933)

- If $k(n) = o(n)$, then the set of $(\ell; k)$-within-perfect numbers has density 0.
- If $k(n) \sim cn$ for some $c > 0$, then the set of $(\ell; k)$-within-perfect numbers has density $D(\ell + c) - D(\ell - c)$.
- If $k(n) \asymp n$, then the set of $(\ell; k)$-within-perfect numbers has positive lower density and upper density strictly less than 1.
- If $n = o(k(n))$, then the set of $(\ell; k)$-within-perfect numbers has density 1.

Better Understanding?

- For the sublinear regime, from the above theorem we only know the density of $(\ell; k)$-within-perfect numbers is 0.

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- For the sublinear regime, from the above theorem we only know the density of $(\ell; k)$-within-perfect numbers is 0.
- The next step is to find an explicit upper bound for the sublinear regime?

Theorem(Cohen-Cordwell-Epstein-K.-Lott-Miller)

Suppose $k(y) \leq y^{\epsilon}$ for large y and k is a positive increasing unbounded function. Consider the following set

$$\Sigma := \left\{ \frac{\sigma(m)}{m} : m \geq 1 \right\} \subset \mathbb{Q}.$$

- If $\ell \in \Sigma$, then for $\epsilon \in (0, 1/3)$ we have

$$\lim_{x \to \infty} \frac{\# W(\ell; k; x)}{x / \log x} = \sum_{\sigma(m) = \ell m} \frac{1}{m}$$

- If $\ell \in (\mathbb{Q} \cap [1, \infty)) \setminus \Sigma$, $\ell = a/b$, $a > b \geq 1$, a, b are coprime integers and $\epsilon \in (0, 1/3)$, then we have the following upper bound

$$\# W(\ell; k; x) = O(\max\{a, b^3\} x^{\min\{3/4,\ \epsilon + 2/3\} + o(1)}).$$

Sketch of Proof

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is a direct consequence of the Prime Number Theorem and our lemma.

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is a direct consequence of the Prime Number Theorem and our lemma.
- Now we want to show
$$\limsup_{x\to\infty} \frac{\#W(\ell; k; x)}{x/\log x} \leq \sum_{\sigma(m)=\ell m} \frac{1}{m}$$

- It suffices to consider $k(y) = y^{\epsilon}$. Fix a large x and let $n \leq x$ satisfy $|\sigma(n) - \ell n| < x^{\epsilon}$.
- Rewrite this Diophantine inequality as a collection of Diophantine equations over certain range, i.e.,

$$\sigma(n) - \ell n \ = \ k, \text{ where } k \in \mathbb{Z}, \ |k| < x^{\epsilon}.$$

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$$\sigma(n) - \ell n \ = \ k, \text{ where } k \in \mathbb{Z}, \ |k| < x^{\epsilon}.$$

- In particular, we have a collection of congruences in the form of regular solutions:

$$\sigma(n) \ \equiv \ k \pmod{n}, \text{ where } k \in \mathbb{Z}, \ |k| < x^{\epsilon}.$$

Recall from Pomerance's Theorems

n is a regular solution if n is of the form

$$n = pm \text{ where } p \text{ is prime, } p \nmid m,\ m \mid \sigma(m), \text{ and } \sigma(m) = k \ (*)$$

We can make the following assumptions one by one:

- n is in the form of (*).

  By Pomerance's theorem, the number of elements of $W(\ell; k; x)$ *NOT* of the form (*) is at most

  $$2x^{\epsilon}x^{2/3+o(1)} = 2x^{2/3+\epsilon+o(1)},$$

  which is negligible (compared with $x/\log x$).

Theorem 4.2 (Hornfeck-Wirsing)

The number of multiply perfect numbers less than or equal to x is at most $x^{o(1)}$ as $x \to \infty$.

- $p > x^\epsilon$ in (*).

  By the Prime Number Theorem and Hornfeck-Wirsing theorem, the number of $n \le x$ of the form (*) with $p \le x^\epsilon$ is at most

$$\frac{x^\epsilon}{\log x^\epsilon} x^{o(1)} \ll_\epsilon \frac{x^{\epsilon+o(1)}}{\log x},$$

  which is again negligible.

- $\sigma(m)/m \leq \ell$ in (*).

  If $\sigma(m) = rm$ for some $r \geq \ell + 1$, then

$$\begin{aligned} \sigma(n) - \ell n &= \sigma(p)\sigma(m) - \ell pm \\ &= (1+p)(rm) - \ell pm \\ &= m(r + p(r - \ell)) \\ &\geq p > x^{\epsilon}. \end{aligned}$$

  Contradiction!

- $\sigma(m)/m = \ell$ in (*).

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  Consider the case where $\sigma(m) = rm$ with $2 \leq r \leq \ell - 1$ and $p > x^{\epsilon}$. Note that $r + p(r - \ell) \geq 0$ implies $p < r \leq \ell - 1$. For $x > (2\ell)^{1/\epsilon}$, we have a contradiction. Now suppose that $r + p(r - \ell) < 0$. Then $|\sigma(n) - \ell n| < x^{\epsilon}$ if and only if $m[(\ell - r)p - r] < x^{\epsilon}$. By Merten's estimate, the number of such n is

$$\begin{aligned}
&\leq \sum_{2 \leq r \leq \ell - 1} \sum_{x^{\epsilon} < p \leq x} \frac{x^{\epsilon}}{(\ell - r)p - r} \\
&\leq (\ell - 2)x^{\epsilon} \sum_{x^{\epsilon} < p \leq x} \frac{1}{p - \ell + 1} \\
&\leq 2(\ell - 2)x^{\epsilon} \sum_{x^{\epsilon} < p \leq x} \frac{1}{p} \\
&\ll (\ell - 2)x^{\epsilon} \log\log x.
\end{aligned}$$

Thus, we only have to work with

$$n = pm \text{ where } p \text{ is prime, } p \nmid m,\ \sigma(m) = \ell m \quad (**)$$

Next we estimate the contribution from (**).

By partial summation and Hornfeck-Wirsing Theorem, we have for any $z \geq 1$,

$$\sum_{\substack{m \leq z \\ \sigma(m) = \ell m}} \frac{\log m}{m} = \int_1^z \frac{\log t}{t} dP(t) = \frac{\log z}{z^{1-o(1)}} + \int_1^z \frac{\log t}{t^{2-o(1)}}\ dt \ \ll 1,$$

where $P(z) = \#\{m \leq z : \sigma(m) = \ell m\}$.
From these we can see that both of the series

$$\sum_{\sigma(m) = \ell m} \frac{\log m}{m}, \quad \sum_{\sigma(m) = \ell m} \frac{1}{m}$$

converge.

For $m \leq x^{\epsilon}$, since

$$0 < \frac{\log m}{\log x} \leq \epsilon < 1,$$

we have

$$\left(1 - \frac{\log m}{\log x}\right)^{-1} = 1 + O_{\epsilon}\left(\frac{\log m}{\log x}\right).$$

Let c be any constant greater than 1. By the Prime Number Theorem, there exists $x_0 = x_0(c) > 0$ such that for $x \geq x_0$, we have

$$\pi(x) < c \ \frac{x}{\log x}.$$

Then for $x \geq \max\{x_0^{1/(1-\epsilon)}, \ell^2\}$, we have

$$\begin{aligned}
\#W(\ell; k; x) &\leq \sum_{\substack{m \leq x^\epsilon \\ \sigma(m) = \ell m}} \pi(x/m) \\
&< c \sum_{\substack{m \leq x^\epsilon \\ \sigma(m) = \ell m}} \frac{x/m}{\log(x/m)} \\
&= c \ \frac{x}{\log x} \sum_{\substack{m \leq x^\epsilon \\ \sigma(m) = \ell m}} \frac{1}{m} + O_\epsilon\Bigg(\frac{cx}{(\log x)^2} \sum_{\substack{m \leq x^\epsilon \\ \sigma(m) = \ell m}} \frac{\log m}{m} \Bigg) \\
&< c \ \frac{x}{\log x} \sum_{\sigma(m) = \ell m} \frac{1}{m} + O_\epsilon\Bigg(\frac{cx}{(\log x)^2} \Bigg).
\end{aligned}$$

Therefore,

$$\limsup_{x\to\infty} \frac{\#W(\ell;k;x)}{x/\log x} \leq \ c \sum_{\sigma(m)=\ell m} \frac{1}{m}.$$

Since the choice of constant $c > 1$ is arbitrary, we have

$$\limsup_{x\to\infty} \frac{\#W(\ell;k;x)}{x/\log x} \leq \ \sum_{\sigma(m)=\ell m} \frac{1}{m}.$$

This completes the proof.

Moments of Thoughts

- Erdös' philosophy: We don't have to count everything. We should figure out where is the major contribution and count (by making more and more assumptions), for the rests simply use trivial bounds.

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- Erdös' philosophy: We don't have to count everything. We should figure out where is the major contribution and count (by making more and more assumptions), for the rests simply use trivial bounds.
- Divide-and-conquer argument
- Of course, our argument is just a simple one! I encourage you to read the papers by Erdös, Pomerance, Pollack etc. to feel their magic and ingenuity.
- Pollack-Pomerance-Thompson's new result is an interesting one and it appears after our paper is submitted for publication. But unfortunately their result lacks uniformity in ℓ and hence cannot be applied to the theorem we just proved.

Another Notion that We Studied

Definition 4.3 (k-near-perfect number)

Let $k \in \mathbb{N}$ be fixed. $n \in \mathbb{N}$ is k-near-perfect if it is the sum of all of its proper divisors with at most k exceptions.

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- For example: Perfect numbers
- 12 is a 1-near-perfect number: $12 = 1 + 2 + 3 + 6$, while all of its proper divisors are $1, 2, 3, 4, 6$.
- 30 is a 4-near-perfect number: $30 = 5 + 10 + 15$, while all of its proper divisors are $1, 2, 3, 5, 6, 10, 15$

Known result

Theorem 4.4 (Pollack-Shevelev 2012)

Let $k \in \mathbb{N}$ and $N(k;x)$ denotes the set of all k-near-perfect numbers up to x. Then as $x \to \infty$,

$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{O_k(1)},$$

where $O_k(1)$ is between $(\log k)/(\log 2) - 3$ and $k - 1$.

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Denote by $N(k;x)$ the set of all k-near-perfect numbers up to x and let $y = x^{\frac{1}{\log\log x}}$.

They partition $N(k;x)$ into

$$
\begin{aligned}
N_1(k;x) &:= \{n \in N(k;x) : n \text{ is } y\text{-smooth}\} \\
N_2(k;x) &:= \{n \in N(k;x) : P^+(n) > y \text{ and } P^+(n)^2 | n\} \\
N_3(k;x) &:= \{n \in N(k;x) : P^+(n) > y \text{ and } P^+(n) || n\},
\end{aligned}
$$

where n is y-smooth if all of its prime factors are at most y and $P^+(n)$ is the largest prime factor of n.

$N_1(k;x)$ and $N_2(k;x)$

Theorem 4.5

Let $u = \log x / \log y$ and $\Phi(x,y)$ be the set of all y-smooth numbers up to x. Then uniformly for $(\log x)^3 \leq y \leq x$, we have

$$\#\Phi(x,y) \; = \; x \exp\left(-u \log u + O(u \log\log u)\right).$$

By the smooth number bound, we have

$$\#N_1(k;x) \leq \Phi(x,y) \ll x \exp(-(\log_2 x)(\log_3 x) + O(\log_2 \log_4 x)),$$

which is negligible.
We also have the following trivial estimate:

$$\#N_2(k;x) \leq \sum_{p>y} \frac{x}{p^2} \ll \frac{x}{y} \; = \; x^{1-1/\log\log x} = x \exp(-\log x / \log\log x),$$

which is negligible.

$N_3(k;x)$

For $n \in N_3(k;x)$, we can write

$$n = pm, \quad \text{where} \quad p = P^+(n) > \max\{y, P^+(m)\}.$$

Further partition $N_3(k;x)$ according to $\tau(m) \leq k$ and $\tau(m) > k$, where $\tau(m)$ is the number of positive divisors of m.

A moment of thought

- By observing trivially that $6p_1 \cdots p_s = p_1 \cdots p_s + 2p_1 \cdots p_s + 3p_1 \cdots p_s$ is $(2^{s+2} - 4)$-near-perfect, we know that

$$\#N(k;x) \gg_k \frac{x}{\log x} (\log \log x)^{\lfloor \frac{\log(k+4)}{\log 2} \rfloor - 3},$$

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- However, the upper bound $\frac{x}{\log x}(\log\log x)^{k-1}$ essentially comes from the trivial count of

$$\#N_3'(k;x) = \{n \le x : n = pm, p = P^+(n) > \max\{y, P^+(m)\},$$
$$\tau(m) \le k, n \text{ is } k\text{-near-perfect}\}.$$

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$$\#N_3'(k;x) = \{n \le x : n = pm, p = P^+(n) > \max\{y, P^+(m)\}, \tau(m) \le k, n \text{ is } k\text{-near-perfect}\}.$$

in which we only capture the information $\tau(m) \le k$. Therefore, the upper bound should be rough.

- The reason we want $\tau(m) > k$ is to ensure

$$\sigma(m) - \sum_{d \in D_n^{(1)}} d > 0.$$

- The reason we want $\tau(m) > k$ is to ensure

$$\sigma(m) - \sum_{d \in D_n^{(1)}} d > 0.$$

- We partition $N_3(k;x)$ differently from Pollack and Shevelev:

$$\begin{aligned} N_3^{(1)}(k;x) &:= \{n \in N_3(k;x) : \text{all of the positive divisors of} \\ &\qquad m \text{ are redundant divisors of } n\}, \\ N_3^{(2)}(k;x) &:= N_3(k;x) \setminus N_3^{(1)}(k;x). \end{aligned}$$

This leads to

Theorem (Cohen-Cordwell-Epstein-K.-Lott-Miller)

For $4 \leq k \leq 9$, we have

$$\#N(k;x) \ \sim \ c_k \frac{x}{\log x}$$

as $x \to \infty$, where

$$c_4 = c_5 = \frac{1}{6}, \ c_6 = \frac{17}{84}, \ c_7 = c_8 = \frac{493}{1260}, c_9 = \frac{179017}{360360}.$$

The new partition is advantageous for the general case. We can handle the general case by doing induction and from the second step onwards, we use sieve estimates rather than smooth number estimates. By a more elaborate argument, we have

Theorem (Cohen-Cordwell-Epstein-K.-Lott-Miller)

For $k \geq 4$, as $x \to \infty$

$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{j_0(k)},$$

where $j_0(k)$ is the smallest integer such that

$$j_0(k) > \frac{\log(k+1)}{\log 2} - \frac{\log 5}{\log 2}.$$

Remark: $\frac{\log 5}{\log 2} \approx 2.3219$.

Theorem (Cohen-Cordwell-Epstein-K.-Lott-Miller)

Let f be the following function defined for integers $k \geq 4$.

$$f(k) = \left\lfloor \frac{\log(k+4)}{\log 2} \right\rfloor - 3.$$

For integer $k \in [4, \infty)_{\mathbb{Z}} \setminus (\{10, 11\} \cup \{2^{s+2} - i : s \geq 3, i = 5, 6\})$, we have

$$\#N(k; x) \asymp_k \frac{x}{\log x} (\log \log x)^{f(k)}.$$

More Results—Generalizations

- We study the following natural generalization:

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Definition 4.6 (k-near-perfect number)

Let $k : [1, \infty) \to \mathbb{R}$ be a positive and increasing function. $n \in \mathbb{N}$ is said to be a k-near-perfect number if n is a sum of all of its proper divisors with at most $k(n)$ exceptions.

Refer to our paper: *On within-perfectness and near-perfectness*, https://arxiv.org/pdf/1610.04253.pdf.

- Thank you and Questions?

Thanks

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- Far-reaching conjecture (even the lower bound)! We think this is the correct one based on some probabilistic heuristics:

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  2. (Cramer's model) The 'probability' of a random integer n being a prime is $1/\log n$. (Based on PNT)
  3. (Kubilius' model) The 'probability' of a random integer to be divisible by a prime p is $1/p$.
- We think of the 'events' of two random integers 'being a prime' are independent. The same for divisibility. (Of course, this is not quite the case. We will do suitable 'corrections'.)